

String from Large N Gauge Fields via Graph Summation on a P^+ - x^+ Lattice^{*†}

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Abstract

I describe renewed efforts to establish a string description of large N_c QCD by summing large “fishnet” diagrams. Earlier work on fishnets indicated that the usual relativistic (zero thickness) string theory can arise at strong 't Hooft coupling, at best yielding a highly idealized description, which fails to incorporate such salient features of continuum QCD as asymptotic freedom and point-like constituents. The recently conjectured AdS/CFT correspondence is compatible with such limitations because it also gives a simple picture of large N_c gauge theory only at strong coupling. In order to better understand how string theory could emerge from large N_c QCD at strong coupling, Klaus Bering, Joel Rozowsky, and I have developed an improved implementation of my effort of the late seventies to digitize the planar diagrams of large N_c light-cone quantized QCD by discretizing both P^+ and x^+ . This discretization allows a strong coupling limit of the sum of planar diagrams to be defined and studied. It also provides a natural framework to explore the possible dual relationship between QCD in light-cone gauge and string theory quantized on the light-cone.

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1 Introduction

It has long been thought that 't Hooft's $N_c \rightarrow \infty$ limit [1] of $SU(N_c)$ gauge theory might be usefully described by some sort of string theory. However, there is an apparently devastating argument, that this “QCD String” (a.k.a. a tower of glueballs) is *not* fundamental string (of “string theory”): the graviton appears in the spectrum of the latter, contradicting the well-known folk theorem [2] forbidding massless spin 2 bound states in a Poincaré invariant quantum field theory[§].

A way to evade this argument has been shown by the conjectured equivalence between classical IIB superstring theory on an AdS_5 background and $\mathcal{N} = 4$ supersymmetric $SU(\infty)$ Yang-Mills on flat 4 dimensional Minkowski space-time [7–9]. The point is that in this example the graviton lives in 5 space-time dimensions and the flat space-time global symmetry (Poincaré(3,1)) is only a subgroup of Poincaré(4,1), which is realized locally, not globally. Thus the “massless” 5 dimensional graviton is a composite of the quanta of a flat-space quantum field theory in 4 dimensional space-time. There is no massless spin 2 particle in this 4d quantum field theory and no folk theorems are violated. In the $\mathcal{N} = 4$ conformally invariant example, the projection of the 5d graviton onto the 4d Minkowski boundary of AdS_5 is a multi-gluon continuum state. But if the mechanism can be extended to the non-conformally invariant gauge theory describing the gluon sector of QCD, the graviton 4d remnant would presumably be a massive spin 2 glueball.

To move these statements beyond conjecture, one clearly needs to establish the “dual” description starting from either of the supposedly equivalent theories. I think it is clear that the best starting point for such a project is the flat space quantum field theory. Unlike the previous conjectured dualities in string theory, which asserted the equivalence of pairs of poorly defined theories, this duality asserts the equivalence of a poorly defined theory (string or quantum gravity) to a perfectly well defined theory (asymptotically free or conformally invariant quantum gauge theory on flat 4d space-time). Indeed, I am inclined to regard this “duality” as more analogous to the alternate descriptions of superconductivity given by BCS theory on the one-hand and Landau-Ginzburg theory on the other. If this metaphor holds, the flat space quantum field theory should be embraced as the long sought microscopic formulation of string/quantum gravity.

As 't Hooft showed in his pioneering paper [1], the $N_c \rightarrow \infty$ limit of $SU(N_c)$ Yang-Mills theory reduces to a certain sum of planar Feynman diagrams. Elegant techniques, involving the explicit elimination of the off-diagonal matrix elements of the matrix field, have been used to obtain this limit in matrix theories of extremely low space-time dimension (namely $D=0,1$) [10], but these methods have failed to deal with theories with space-time dimension $D > 1$. At the moment, I see no better approach to the $D > 1$ case than setting up a framework to carry out the direct summation of planar graphs. In the mid-1970's, motivated by the success of light-cone quantization of string theory [11], I proposed [12] that planar diagram sums be carried out by using light-cone parameterization $x^\pm \equiv (t \pm z)/\sqrt{2}$ and that a convenient way to digitize the sum was to discretize the momentum conjugate to x^- , $P^+ = lm$ with $l = 1, 2, \dots$, and imaginary light-cone time $ix^+ = ka$, with $k = 1, 2, \dots$. In those first papers I restricted attention to scalar field theories, but Brower, Giles, and I soon made a first attempt to extend the approach to QCD [13]. In our setup, the strong 't Hooft coupling limit $N_c g^2 \rightarrow \infty$ favors the fishnet [14] diagrams that lead to a light-cone string

[§]Incidentally, the conceptual problems with the formulation of string theory (and its low energy limit, quantum gravity) would disappear if string (including its graviton) were a composite structure arising in ordinary flat space quantum field theory. Sakharov proposed this idea for quantum gravity long ago [3], and it was later vigorously explored by Adler [4] and Zee [5]. Early in this game, it was realized that the underlying flat space theory cannot be Poincaré invariant, because of this same folk theorem. My suggestion that string be a composite of string bits [6] also evades the theorem because the bits live in at least one less space dimension than string, so the space-time symmetry of the underlying string bit dynamics is only a subgroup of the Poincaré group of string.

interpretation. Of course, by its very nature a strong coupling limit probes the microscopic details of discretization, and can at best show only rough qualitative resemblance to the continuum theory. Even so, there were a number of loose ends and unsatisfactory features of this first discretization of QCD which needed to be addressed.

Motivated by the goal of discovering a more definitive string description of large N_c QCD Klaus Bering, Joel Rozowsky, and I set out to remedy these shortcomings, and in this talk I would like to tell you about the results of our efforts [15]. As you will see, we have obtained a much improved discretization setup, but have just begun to explore its usefulness in capturing a string picture of the sum of planar diagrams.

2 String on a Light-Front

An evolving string sweeps out a world sheet $x^\mu(\sigma^1, \sigma^0)$ in space-time. One can choose the parameters so that $x^+ = \sigma^0$, and so that \mathcal{P}^+ , the density of P^+ , is a constant T_0 . Then evolution in x^+ is generated by the Hamiltonian

$$H \equiv P^- = \int_0^{P^+/T_0} d\sigma \left[\frac{\mathcal{P}^2}{2T_0} + \frac{T_0}{2} \mathbf{x}'^2 \right], \quad (2.1)$$

where for brevity we have called $\sigma^1 = \sigma$ and the prime denotes differentiation with respect to σ . Of course $\mathcal{P}(\sigma)$ is the momentum conjugate to $\mathbf{x}(\sigma)$. A key novelty here is that P^+/T_0 measures the quantity of string present, and its interpretation as a component of momentum is secondary and derivative. In this way the string is seen to enjoy a Galilei invariant dynamics as it moves only in the $d - 2$ dimensional transverse space.

As shown by Mandelstam [16], interactions are easily introduced by using the path history form of quantum mechanics and including histories in which strings break and join. Technically this is accomplished by first obtaining the imaginary time $ix^+ \equiv \tau$ path integral representation of $\langle f | e^{-\beta H} | i \rangle$ for free string. The action in this case is just

$$S^{Free}(\beta) = \frac{T_0}{2} \int_0^\beta d\tau \int_0^{P^+/T_0} d\sigma [\dot{\mathbf{x}}^2 + \mathbf{x}'^2]. \quad (2.2)$$

Here the action is seen to be an integral over a simply connected rectangular domain for an open free string and a cylinder for a closed string. A diagram describing an arbitrary number of splits and joins is obtained by allowing some number of cuts, each at constant σ but of varying length, within the domain. The ends of these cuts mark the splitting or joining points, and \mathbf{x} is discontinuous across them. Calling the generic such domain Σ , the complete amplitude is then given by

$$\mathcal{M} = \sum_{\Sigma} \int \mathcal{D}\mathbf{x} e^{-S(\beta, \Sigma)}. \quad (2.3)$$

3 Discretization

In order to give a nonperturbative definition of the path integrals appearing in the previous section, Roscoe Giles and I introduced a lattice version [17] of the domains Σ . It was only necessary to discretize τ and σ . So we set $\beta = (N + 1)a$ and $P^+ = Mm \equiv MaT_0$, with M and N fixed positive

integers. Then $\mathbf{x}(\sigma, \tau)$ is replaced by \mathbf{x}_{lk} , and the functional integration by ordinary integrals. Finally, the action is simply replaced by

$$S = \frac{T_0}{2} \sum_{L \in \Sigma} \Delta \mathbf{x}_L^2, \quad (3.1)$$

where L labels a link on the lattice. Links between nearest neighbor sites in the τ direction are all present, but those in the σ direction are occasionally absent reflecting the possibility of splits and joins. It is a highly nontrivial fact that this apparently noncovariant setup turns out, after the continuum limit, to be fully compatible with Poincaré invariance in the critical dimension.

4 Feynman Diagrams on a Light-Front

To understand how Feynman diagrams look in light-cone parameterization, consider the mixed representation of the scalar field propagator:

$$\Delta(\mathbf{p}, p^+, ix^+) = \theta(x^+ p^+) \frac{e^{-ix^+(\mathbf{p}^2 + \mu^2)/2p^+}}{2|p^+|}. \quad (4.1)$$

For simplicity we may establish the convention that each line propagates forward in x^+ , and correspondingly $p^+ > 0$. Also we may pass to imaginary time, $\tau = ix^+ > 0$ and then write

$$\Delta(\mathbf{p}, p^+, \tau) = \frac{e^{-\tau(\mathbf{p}^2 + \mu^2)/2p^+}}{2p^+}, \quad \tilde{\Delta}(\mathbf{x}, p^+, \tau) = \left(\frac{p^+}{2\pi\tau} \right)^{d/2} \frac{e^{-p^+ \mathbf{x}^2 / 2\tau - \mu^2 \tau / 2p^+}}{2p^+} \quad (4.2)$$

where the second form is in transverse coordinate representation and $d = D - 2$ is the dimensionality of transverse space.

Discretizing $\tau = ka$ and $p^+ = lm = laT_0$, $k, l = 1, 2, \dots$, the coordinate propagator becomes[¶]

$$\tilde{\Delta}_{lk}(\mathbf{x}) = \left(\frac{lT_0}{2k\pi} \right)^{d/2} \frac{e^{-T_0(l/2k)\mathbf{x}^2 + k\mu^2/2lT_0}}{2lm}. \quad (4.3)$$

Comparing to the previous section, we see that we can crudely think of a planar Feynman diagram as a (coarsely) discretized world sheet, with a dynamical link dependent string tension $T_{lk} = (l/k)T_0$. Each link has its own independent k, l which are each summed over all positive integers. Of course only the large fishnet diagrams will bear any actual resemblance to a continuous world sheet! The sum over all planar diagrams would then define the QCD string dynamics as including an average over all such discretizations ranging from coarse to fine. Also note that a good world sheet path integral should have (effectively) only positive weights. For instance, for $\lambda\phi^4$ theory each vertex contributes a minus sign if $\lambda > 0$. A good world sheet interpretation requires $\lambda < 0$, the attractive unstable sign. Gauge theories have vertices of both signs, complicating a straightforward world sheet interpretation.

To illustrate the effect of our discretization on a standard diagrammatic calculation, consider the sum of those 2 to 2 scattering diagrams in $\lambda\phi^4$ theory shown in Fig. 1. We work in the transverse center of mass frame. Assume that the (discrete) P^+/m of the initial (final) particles is $l, M - l$ ($r, M - r$). Let $E = M\mathbf{p}^2/2l(M - l)$ be the initial energy (P^-). Fix the discrete time of

[¶]The discretization of τ provides a universal ultraviolet cutoff, and every diagram will therefore be finite. In contrast, the DLCQ industry [18] keeps time continuous, and must regulate ultraviolet divergences in some other fashion.

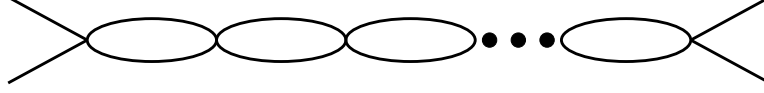


Figure 1: Diagrams of ϕ^4 scalar field theory summed in the text.

the first vertex at 0, and sum over all diagrams in which the final particles both propagate to time k , multiply by e^{kaE} and sum over all k from 1 to ∞ . The result for the off-energy shell S-matrix is then

$$S_{fi} = \delta_{ri}\delta(\mathbf{p} - \mathbf{p}') + \frac{-\lambda}{32\pi^3 \sqrt{l(M-l)r(M-r)}} \left(e^{-aE + M\mathbf{p}'^2/2r(M-r)T_0} - 1 \right)^{-1} \left(1 - \frac{\lambda(M-1)}{16\pi^2 M} \ln(1 - e^{aE}) \right)^{-1} \quad (4.4)$$

Compared to standard formal scattering theory, we see that instead of a factor $1/(E_f - E - i\epsilon)$ which acts in wavepackets like the standard energy conserving delta function, the use of discrete time has rendered this as $1/(e^{aE_f - aE} - 1)$. This replacement is easy to understand because with discrete imaginary time, the amplitudes should be periodic in E with period $2\pi i/a$. It is thus apparent that the scattering amplitude should be identified with the coefficient of this factor. For $\lambda < 0$, the scattering amplitude shows a bound state pole at a real negative value of E :

$$E_B \equiv -B = \frac{1}{a} \ln \left[1 - e^{16\pi^2 M/\lambda(M-1)} \right].$$

In the continuum limit, $M \rightarrow \infty, a \rightarrow 0$, one can make the pole location stay finite by tuning λ to vanish logarithmically as $a \rightarrow 0$, showing dimensional transmutation in an asymptotically free theory.

5 Discretization Setup for Yang-Mills Field Theory

The discretization of QCD initially attempted by Brower, Giles, and me [13], was based on a literal transcription of the Feynman rules in light-cone gauge. The transverse gauge field can be described in the complex basis $(A_1 \pm iA_2)$ when it takes on the guise of a complex scalar field, described diagrammatically by attaching an arrow to each line of a Feynman diagram. The primitive quartic vertex conserves arrows, but the cubic vertices can act as sources or sinks of arrows. The longitudinal gauge field A_+ does not propagate and can be integrated out to yield an induced quartic vertex, which depends upon the P^+ values of the incoming legs in a singular way:

$$\Gamma_{\text{induced}}^4 = g^2 \frac{(P_1^+ + P_1^{+'})(P_2^+ + P_2^{+'})}{(P_1^+ - P_1^{+'})^2}. \quad (5.1)$$

Upon discretization, we adopted the drastic prescription of simply dropping the infinite contribution at $P_1^{+'} = P_1^+$. Furthermore all tadpole diagrams had to be dropped, because of our insistence that no line propagate 0 time steps. Then the strong coupling limit singled out large planar diagrams involving only the primitive quartic couplings, thus leading to an evaluation similar to the ϕ^4 example of the previous section. Unfortunately, these quartic couplings have mixed signs: an attractive interaction between gluons of parallel spin and repulsive between gluons of antiparallel spin. This ferromagnetic interaction pattern meant that our discretization led to a formal strong

coupling limit in which the only long string that could form in the limit would have huge total spin. The essential problem is that attractive interactions between gluons of opposite spin arise in QCD from gluon exchange [19], and the discretization chosen in [13] prevents anti-ferromagnetic gluon exchange from competing at strong coupling with the ferromagnetic quartic interaction. This, together with our drastic prescription for all of the $P^+ = 0$ ills of light-cone quantization, points to the need for a more refined discretized model to adequately describe the strong coupling behavior of large N_c QCD.

5.1 Improved Discretization Rules

Clearly what is needed is a prescription that either enhances strong coupling gluon exchange or suppresses the strong coupling quartic interactions. We found it most natural to arrange the latter by abolishing all quartic interactions, primitive and induced, and replacing them with the exchange of short lived fictitious particles. This is shown for the primitive quartic interaction in Fig.2. The

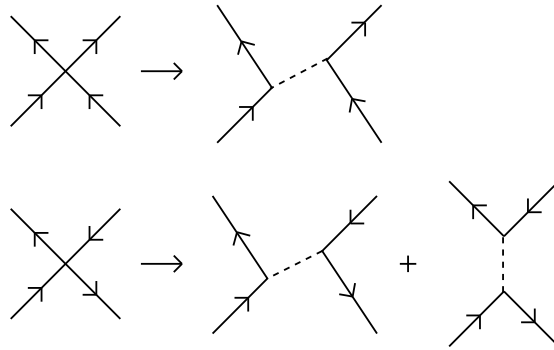


Figure 2: Quartic vertices get replaced by two cubic vertices and a fictitious scalar field.

dashed line is associated with the fictitious two-form propagator

$$\Delta_{2\text{-form}} = -h_k e^{-k\mathbf{Q}^2/2lT_0}, \quad \sum_{k=1}^{\infty} h_k = 1, \quad (5.2)$$

where the h_k are tunable parameters which are required to vanish rapidly with $k = 1, 2, \dots$, the number of time steps propagated. We treat the induced quartic interaction in a similar fashion, giving the non-dynamical field A_+ a short-time propagator

$$\Delta_+ = -f_k e^{-k\mathbf{Q}^2/2lT_0}, \quad \sum_{k=1}^{\infty} f_k = 1. \quad (5.3)$$

The presence of the tunable parameters f_k and h_k is very welcome, because they can be adjusted to arrange the cancelation of cut-off artifacts that can typically spoil Poincaré invariance in the continuum limit. As a bonus, we find that our prescription provides the appealing interpretation of tadpole diagrams indicated in Fig. 3. The first two diagrams cancel exactly (which is fortunate since they can't be drawn in our discrete model!) leaving the third diagram which *can* be drawn. Our complete set of Feynman rules is neatly summarized in Fig. 4 taken from our paper [15]. Note that to avoid clutter we have suppressed the double line notation so these rules are completely sufficient for all graphs of planar or cylindrical topology ($N_c \rightarrow \infty$). For general diagrams, the double line notation must be restored in order to properly account for the $1/N_c$ corrections.

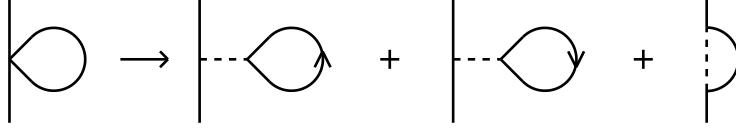


Figure 3: Three tadpole diagrams resulting from the spreading out of the quartic gauge vertex.

	0		$\frac{g}{T_0}(M_2 - M_1)$
	$-2\frac{g}{T_0}\left(\frac{M_1+M_2}{M_1M_2}\right)(M_1Q_2^\wedge - M_2Q_1^\wedge)$		$+\frac{g}{T_0}$
	$-2\frac{g}{T_0}\left(\frac{M_1+M_2}{M_1M_2}\right)(M_1Q_2^\vee - M_2Q_1^\vee)$		$-\frac{g}{T_0}$
	$\frac{1}{2M}e^{-k\mathbf{Q}^2/2MT_0}$		
	$-f_k\frac{T_0}{M^2}e^{-k\mathbf{Q}^2/2MT_0}$		
	$-h_kT_0e^{-k\mathbf{Q}^2/2MT_0}$		

Figure 4: Summary of discretized Feynman rules using only cubic vertices. We have explicitly inserted a factor of $1/T_0$ for each vertex arising from the discretization.

An easy way to understand why a set of rules with only cubic vertices is possible, is to apply light-cone gauge to the Yang-Mills Lagrangian in first order form. The upshot is the Lagrangian

$$\begin{aligned}
\mathcal{L} = & \frac{1}{2}\text{Tr}\partial A_k \cdot \partial A_k + \frac{1}{2}\text{Tr}\hat{A}^2 + \text{Tr}\phi_{kl}^2 - 2ig\text{Tr}\frac{1}{\partial_-}A_k[\partial_-A_n\partial_kA_n - \partial_kA_n\partial_-A_n] \\
& - ig\text{Tr}\frac{1}{\partial_-}\hat{A}[A_k, \partial_-A_k] - ig\text{Tr}\phi_{kn}[A_k, A_n],
\end{aligned} \tag{5.4}$$

where $\hat{A} \equiv \partial_-A_+ - \nabla \cdot \mathbf{A}$. This makes it clear why we called the fictitious scalar represented by the dashed lines a 2-form: it is a (pseudo) scalar only in $3 + 1$ dimensions.

5.2 One Loop Self Energy

Quite apart from its use as a facilitator for a strong coupling expansion, our discretization can also serve as a novel way to regulate the diagrams of weak coupling perturbation theory. To illustrate this aspect, we quote the result for the one loop gluon self-energy, with discretization in place:

$$\Pi_2(Q^2) = \frac{g^2N_c}{4\pi^2}\left[\sum_{k=1}^{\infty}\frac{u^k}{k^2}\left(4M[\psi(M) + \gamma] - \frac{(M-1)(11M-1)}{3M}\right)\right]$$

$$-\sum_{k=1}^{\infty} u^k \sum_{l=1}^{M-1} \frac{lh_k(l)}{Mk} - \sum_{k=1}^{\infty} f_k \frac{u^k}{k} \left(4M[\psi(M) + \gamma] - \frac{7(M-1)}{2} \right) \Big], \quad (5.5)$$

where $u = e^{-Q^2/2MT_0}$. In order to cancel lattice artifacts in the continuum limit $M \rightarrow \infty$, we find the constraints

$$\sum_{k=1}^{\infty} \frac{f_k}{k} = \frac{\pi^2}{6}, \quad \sum_{k=1}^{\infty} \frac{h_k(l)}{k} = -\frac{\pi^2}{18} \left(1 - \frac{1}{l} \right). \quad (5.6)$$

Then we obtain

$$\Pi_2 \rightarrow \frac{g^2 N_c}{16\pi^2} \frac{Q^2}{T_0} \left\{ \left[8(\ln M + \gamma) - \frac{22}{3} \right] \ln \frac{Q^2}{2MT_0} + \frac{4}{3} \right\}. \quad (5.7)$$

Here $2MT_0$ functions as a uv cut-off. With this understanding our result for Π_2 agrees exactly with the known light-cone gauge result [20]. Notice that the parameters f_k, h_k which specify our discretization enter weak coupling physics only through their *moments*, for example $\sum_k f_k/k$. In contrast, the strong coupling limit is sensitive to the values of these parameters at low k . Thus the two limits give complementary constraints on these parameters.

6 Concluding Remarks

Our main aim in developing this discretization formalism is to establish a framework for handling the sum of all the planar diagrams of $N_c \rightarrow \infty$ gauge theories. As yet we don't have any dramatic results to report. However we have begun studying how the machinery works in simpler situations than full-blown gauge theories. In our paper [15] we worked out the sum of the densest (strong coupling "fishnet") planar diagrams of $\text{Tr}\phi^3$ scalar field theory. As expected the result leads to the light-cone quantized bosonic string (with all of its usual pathologies, including the tachyon). The presence of tachyons is not particularly surprising, since the energy density of the theory is unbounded below. We have not made analogous progress on the corresponding diagrams of gauge theories. But we have studied the latter theory in the sectors with $M = 2$. This is not particularly difficult, since the limitation on M reduces the sum of all possible diagrams to a geometric series. Nevertheless, it is interesting, for example, that the $M = 2$ gluon propagator summed to all orders in perturbation theory displays no additional poles beyond that of the massless gluon itself. In contrast the $M = 2$ propagator for the fictitious 2-form field shows a bound state pole at sufficiently strong coupling $g^2 N_c / 8\pi^2 > 18.28$. This indicates that the 2-form field may be particularly important for the understanding of the strong coupling limit.

Another relatively simple testing ground for our formalism is quantum field theory in low space-time dimensions, the simplest being the 't Hooft model (QCD in one space and one time dimensions). Rozowsky and I are just finishing up a study of this model. Since this model is well understood even in the continuum limit, we used it mainly as a test of how our model approaches the continuum theory. One interesting feature of our simultaneous discretization of P^+ and x^+ is that one can approach the continuum in different directions. For example the approach to continuum at fixed $T_0 \equiv m/a$ is different from the approach studied in conventional DLCQ. The latter keep x^+ continuous throughout (in our language this means taking $a \rightarrow 0$ *first* followed by $m \rightarrow 0$). We have confirmed that the same continuum physics emerges in both cases.

I would like to conclude this talk with some remarks on longer term prospects and goals. Our renewed efforts to sum planar graphs have been directly stimulated by the ADS/CFT duality

[7] proposed in the last couple of years. This duality in turn was recognized [8, 9] to be a higher dimensional realization of 't Hooft's concept of holography [21]: the vision that a consistent quantum theory of gravity requires our apparently 3 dimensional spatial world to be 2 dimensional. I have advocated string bits as a way to realize holography in 't Hooft's original 2 dimensional sense: the two dimensions being the transverse dimensions of light-cone string. However, the QCD gluons of this talk really live in 3 dimensions in spite of their description on the light-cone: the longitudinal dimension hasn't disappeared. Rather, it has been disguised as a variable Newtonian mass. (In the ADS/CFT duality this third dimension gets interpreted as a fifth dimension, whence holography is the statement that a 4+1 dimensional effective theory arises from a 3+1 dimensional quantum field theory.) The defining character of string bits is that they have a *fixed* Newtonian mass, in sharp contrast to the gluons we have been describing. To understand 3+1 gauge theories as part of a string bit theory, a gluon with $P^+ = lm$ must in reality be a composite of l string bits: the gluon vertices would then be effective fission/fusion amplitudes as in nuclear physics, rather than fundamental interactions.

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References

- [1] G. 't Hooft, *Nucl. Phys.* **B72** (1974) 461.
- [2] S. Weinberg and E. Witten, *Phys. Lett.* **96B** (1980) 59.
- [3] A. D. Sakharov, *Dok. Akad. Nauk. SSSR* **177** (1967) 70 [*Sov. Phys. Dokl.* **12** (1968) 1040].
- [4] S. L. Adler, *Phys. Rev.* **D14** (1976) 379; *Phys. Rev. Lett.* **44** (1980) 1567; *Phys. Lett.* **95B** (1980) 241; *Rev. Mod. Phys.* **54** (1982) 729.
- [5] A. Zee, *Phys. Rev. Lett.* **42** (1979) 417; *Phys. Rev.* **D23** (1981) 858; *Phys. Rev. Lett.* **48** (1982) 295; *Phys. Lett.* **109B** (1982) 183.
- [6] C. B. Thorn, in *Sakharov Memorial Lectures in Physics*, Ed. L. V. Keldysh and V. Ya. Fainberg, (Nova Science, New York, 1992) 447-454, hep-th/9405069.
- [7] J. M. Maldacena, *Adv. Theor. Math. Phys.* **2** (1998) 231-252, hep-th/9711200.
- [8] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, *Phys. Lett.* **B428** (1998) 105, hep-th/9802109.
- [9] E. Witten, *Adv. Theor. Math. Phys.* **2** (1998) 253-291 hep-th/9802150.
- [10] E. Brezin, C. Itzykson, G. Parisi, and J.-B. Zuber, *Comm. Math. Phys.* **59** (1978) 35.
- [11] P. Goddard, J. Goldstone, C. Rebbi, and C. B. Thorn, *Nucl. Phys.* **B56** (1973) 109.
- [12] C. B. Thorn, *Phys. Lett.* **70B** (1977) 85; *Phys. Rev.* **D17** (1978) 1073.
- [13] R. Brower, R. Giles, and C. Thorn, *Phys. Rev.* **D18** (1978) 484.
- [14] H. B. Nielsen and P. Olesen, *Phys. Lett.* **32B** (1970) 203; B. Sakita and M. A. Virasoro, *Phys. Rev. Lett.* **24** (1970) 1146.

- [15] Klaus Bering, Joel S. Rozowsky, and Charles B. Thorn. *Phys. Rev.*, D61:045007, 2000.
- [16] S. Mandelstam, *Nucl. Phys.* **B64** (1973) 205; *Phys. Lett.* **46B** (1973) 447; *Nucl. Phys.* **B69** (1974) 77; see also his lectures in *Unified String Theories*, ed. M. Green and D. Gross (World Scientific) 1986.
- [17] R. Giles and C. B. Thorn, *Phys. Rev.* **D16** (1977) 366.
- [18] S. J. Brodsky, H-C. Pauli, and S. J. Pinsky, *Phys. Rept* **301**, (1998) 299; hep-ph/9705477.
- [19] C. B. Thorn, *Phys. Rev.* **D59** (1999) 116011, hep-th/9810066.
- [20] C. B. Thorn, *Phys. Rev.* **D20** (1979) 1934.
- [21] G. 't Hooft, “Dimensional Reduction in Quantum Gravity”, in Salamfest 1993, gr-qc/9310026.